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THE EXT CLASS OF AN APPROXIMATELY INNER AUTOMORPHISM

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Abstract. Let A be a simple unital $A\mathbf{T}$ algebra of real rank zero. It is shown below that the range of the natural map from the approximately inner automorphism group to $KK(A,A)$ coincides with the kernel of the map $KK(A,A) \to \bigoplus_{i=0}^{1} \mathrm{Hom}(K_i(A), K_i(A))$.

1. Introduction and preliminaries

1.1. An automorphism α of a unital C*-algebra A is said to be an approximately inner automorphism if there is a sequence of unitaries $u_n \in A$ such that $\alpha(a) = \lim \mathrm{Ad}\, u_n(a)$ for all $a \in A$. It follows that the induced map on $K_*(A)$ is the identity map; there is, however, an invariant of K-theoretical interest which can occur. Nontrivial extensions may arise in the six-term periodic sequence for the K-theory of the mapping torus. We show below that every extension does arise if A is a simple unital $A\mathbf{T}$ algebra of real rank zero. As an immediate corollary we obtain a stronger form of Elliott's classification theorem for such algebras: an invertible KK element that preserves positivity and the class of the unit lifts to an isomorphism.

1.2. Recall that a unital C*-algebra A is said to be a unital $A\mathbf{T}$ algebra if it is expressible as the inductive limit of finite direct sums of algebras of the form $M_n \otimes C(\mathbf{T})$ with unital embeddings. Let A be a unital $A\mathbf{T}$ algebra and let α be an approximately inner automorphism of A. The mapping torus of α is the C*-algebra

$$M_\alpha = \{x \in C[0,1] \otimes A : \alpha(x(0)) = x(1)\}.$$

It will be convenient to identify SA, the suspension of A, with the kernel of the map of evaluation at zero, $e_0 : M_\alpha \to A$. From the short exact sequence:

$$0 \;\to\; SA \;\to\; M_\alpha \;\xrightarrow{e_0}\; A \;\to\; 0$$

one obtains the usual six-term periodic exact sequence:

$$\begin{array}{ccccc} K_0(SA) & \to & K_0(M_\alpha) & \to & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \leftarrow & K_1(M_\alpha) & \leftarrow & K_1(SA) \end{array}$$

In identifying $K_i(SA)$ with $K_{i+1}(A)$ the index maps become $\mathrm{id} - \alpha_*$ (cf. [B, 10.4.1]); further, as α is an approximately inner automorphism one has $\alpha_* = \mathrm{id}$ and the index

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maps are zero. Consequently, the six-term periodic exact sequence reduces to a pair of short exact sequences:

$$0 \rightarrow K_{i+1}(A) \rightarrow K_i(M_\alpha) \rightarrow K_i(A) \rightarrow 0$$

for $i = 0, 1$. Let $\eta_i(\alpha)$ denote the class of this sequence in $\mathrm{Ext}(K_i(A), K_{i+1}(A))$. Since A is an inductive limit of type I C*-algebras, A is in the bootstrap class, and thus the universal coefficient theorem of Rosenberg and Schochet (see [RS, 1.17]) applies: that is, the following is exact:

$$0 \rightarrow \bigoplus_{i=0}^{1} \mathrm{Ext}(K_i(A), K_{i+1}(A)) \xrightarrow{\delta} KK(A, A)$$
$$\xrightarrow{\gamma} \bigoplus_{i=0}^{1} \mathrm{Hom}(K_i(A), K_i(A)) \rightarrow 0.$$

Rørdam observed (see [Rø, p. 435]) that $\delta(\eta_0(\alpha), \eta_1(\alpha)) = 1 - [\alpha]$, where $[\alpha]$ denotes the class of α in $KK(A, A)$. Since the product of any two elements in $\ker \gamma$ is 0 ([RS, 7.10]), one has $\eta_i(\alpha\beta) = \eta_i(\alpha) + \eta_i(\beta)$ for approximately inner automorphisms α, β.

1.3. Since A is a unital $A\mathbf{T}$ algebra, both $K_0(A)$ and $K_1(A)$ are torsion free and so the extensions above are pure (i.e. locally trivial). Given a pair of countable abelian groups P, Q; if $Q = \lim Q_n$ (with connecting maps $f_n : Q_n \rightarrow Q_{n+1}$) one has the following short exact sequence for $\mathrm{Ext}(Q, P)$ (cf. [Ro]):

$$0 \rightarrow \varprojlim{}^1 \mathrm{Hom}(Q_n, P) \rightarrow \mathrm{Ext}(Q, P) \rightarrow \varprojlim \mathrm{Ext}(Q_n, P) \rightarrow 0.$$

If Q is torsion free, the Q_n may be chosen to be isomorphic to $\mathbf{Z}^{k_n}$; in this case $\mathrm{Ext}(Q_n, P) = 0$ and one has the isomorphism

$$\varprojlim{}^1 \mathrm{Hom}(Q_n, P) \cong \mathrm{Ext}(Q, P).$$

In showing that every Ext class arises from the KK-class of an approximately inner automorphism, it will be useful to have an explicit form for this isomorphism. Note that $\varprojlim{}^1 \mathrm{Hom}(Q_n, P)$ may be identified with the cokernel of the map, $\prod_n \mathrm{Hom}(Q_n, P) \rightarrow \prod_n \mathrm{Hom}(Q_n, P)$, where $(g_n) \mapsto (g_n - g_{n+1} f_n)$. Given (g_n), we construct an extension

$$0 \rightarrow P \rightarrow E \rightarrow Q \rightarrow 0$$

as follows: set $E_n = P \oplus Q_n$ and define $h_n : E_n \rightarrow E_{n+1}$ by $h_n(p, q) = (p + g_n(q), f_n(q))$; then $E = \lim E_n$ gives the desired extension. If $P = \lim P_n$ then, by passing to a suitable subsequence (and relabeling), one has $E \cong \lim P_n \oplus Q_n$.

We will have occasion to consider inductive systems of short exact sequences; such an inductive system will consist of a sequence of short exact sequences:

$$0 \rightarrow P_n \rightarrow E_n \rightarrow Q_n \rightarrow 0$$

together with maps between them such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \rightarrow & P_n & \rightarrow & E_n & \rightarrow & Q_n & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & P_{n+1} & \rightarrow & E_{n+1} & \rightarrow & Q_{n+1} & \rightarrow & 0 \end{array}$$

Note that the limit of the inductive system is again a short exact sequence:

$$0 \rightarrow P \rightarrow E \rightarrow Q \rightarrow 0.$$

1.4. In §2 we compute the class of the two extensions arising from the K-theory of the mapping torus for certain approximately inner automorphisms of a unital $A\mathbf{T}$ algebra (see Theorem 2.4). Here we do not assume that the C*-algebra A is simple or has real rank zero. The unitality is not essential either if we define an automorphism to be approximately inner when it is so in the algebra A^+ obtained by adjoining a unit to A, since A^+ is again an $A\mathbf{T}$ algebra (see [E, §1]). The maps that determine the extensions keep track of the K_1-class of $u_{n+1}{u_n}^*$ for the various partial embeddings (this gives the maps $K_0(A_n) \to K_1(A)$) as well as the Bott number for approximately commuting unitaries (see [Ex]): it may be assumed that u_{n+1} approximately commutes with the canonical central unitaries of A_n (this gives the maps $K_1(A_n) \to K_0(A)$).

Definition. Given two unitary $n \times n$ matrices U and V with $||VUV^*U^* - 1|| < 2$, there is a selfadjoint matrix H with $||H|| < 1/2$ such that $VUV^*U^* = e^{2\pi i H}$. Then define the Bott number of the pair by $B(U, V) = \mathrm{Tr}(H)$.

Since $\det(VUV^*U^*) = 1$, it follows that $\mathrm{Tr}(H) \in \mathbf{Z}$. Note that $B(U, V) = \omega(U, V)$, where $\omega(U, V)$ is the winding number of the loop (see [Ex, Lemma 3.1] and [EL]):

$$t \mapsto \det((1-t)UV + tVU) = \det(UV)\det(1 - t + tVUV^*U^*).$$

Thus, the Bott number is invariant under homotopy of pairs of unitaries for which the norm of the commutator is less than 2. Moreover, if $||V_iU{V_i}^*U^* - 1|| < 1$ for $i = 1, 2$, then $B(U, V_1V_2) = B(U, V_1) + B(U, V_2)$.

1.5. We proceed to the main result, Theorem 3.1, in §3. Let A be a simple unital $A\mathbf{T}$ algebra of real rank zero; we show for $i = 0, 1$ that, given

$$\eta \in \mathrm{Ext}(K_i(A), K_{i+1}(A))$$

there is an approximately inner automorphism α for which $\eta_i(\alpha) = \eta$ and $\eta_{i+1}(\alpha) = 0$. The proof is divided into the two cases $i = 0, 1$ and proceeds through a sequence of lemmas; one must for example show that an extension is expressible as an inductive limit of groups of the form $\mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n}$ in a way that makes all relevant diagrams commute.

1.6. With A as above, $K_0(A)$ is a simple dimension group and $K_1(A)$ is a (countable) torsion free group. $K_i(A)$ is thus expressible as the inductive limit of a system

$$\mathbf{Z}^{k_1} \xrightarrow{\chi_1^i} \mathbf{Z}^{k_2} \xrightarrow{\chi_2^i} \mathbf{Z}^{k_3} \to \cdots$$

with $\chi_n^0(j, k) > 0$ for all j, k and n. We may assume that image of $\mathbf{Z}^{k_1}$ contains the class of the unit $[1_A] \in K_0(A)$. We show below that, under mild assumptions on the χ_n^i, A is expressible as an inductive limit of certain maps $\varphi_n : A_n \to A_{n+1}$, where $K_i(A_n) = \mathbf{Z}^{k_n}$ and $(\varphi_n)_* = (\chi_n^0, \chi_n^1)$ and the partial embeddings of φ_n are given by the standard maps described below. These maps are closely related to those arising in the path model constructed by Deaconu in [D].

Let k_0, k_1 be integers with $k_0 \gg 0$. Define $\varphi_{k_0,k_1} : C(\mathbf{T}) \to M_{k_0} \otimes C(\mathbf{T})$ by

$$\varphi_{k_0,k_1}(u)(z) = \begin{pmatrix} 0 & 0 & \cdots & z^{k_1} \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

for $k_1 \neq 0$, and define $\varphi_{k_0,0}$ by

$$\varphi_{k_0,0}(u)(z) = \begin{pmatrix} \varphi_{\lfloor k_0/2 \rfloor,1} & 0 \\ 0 & \varphi_{\lceil k_0/2 \rceil,-1} \end{pmatrix}.$$

Note that when $K_i(C(\mathbf{T}))$ is identified with $\mathbf{Z}$, $(\varphi_{k_0,k_1})_*$ is given by multiplication by k_i. We will refer to φ_{k_0,k_1} as a standard embedding of type (k_0, k_1). Note that under the assumption $k_0 \gg 0$ the image of the unitary generator v of $C(\mathbf{T})$ has small spectral variation in $M_{k_0} \otimes C(\mathbf{T})$; indeed, if $k_1 \neq 0$ the spectrum of $\varphi_{k_0,k_1}(v)(z)$ consists of the k_0^{th} roots of z^{k_1}, where $z \in \mathbf{T}$. To avoid cumbersome notation we let φ_{k_0,k_1} also denote the map $M_n \otimes C(\mathbf{T}) \to M_n \otimes M_{k_0} \otimes C(\mathbf{T})$ (where $x \otimes a \mapsto x \otimes \varphi_{k_0,k_1}(a)$).

A unitary $u \in M_n \otimes C(\mathbf{T})$ is said to be normal if there are integers p, q with $p \neq 0$, $q > 0$ and $q|n$ such that the eigenvalues of u_t (for $t \in \mathbf{T} = \mathbf{R}/\mathbf{Z}$) are $e^{2\pi i(pt+r)/q}$ for $r = 0, 1, \ldots, q-1$, and each eigenvalue occurs with multiplicity n/q. Observe that the spectral variation of u,

$$\sup_{s,t \in \mathbf{T}} \operatorname{dist}(\sigma(u_s), \sigma(u_t)),$$

is π/q, where $\sigma(u)$ denotes the spectrum of the unitary u counted with multiplicity and distance is given by the arc length metric. Note that $\varphi_{q,p}(v) \in M_q \otimes C(\mathbf{T})$ (where v is the unitary generator of $C(\mathbf{T})$) is normal with parameters p, q.

Fact. Suppose $v \in M_n \otimes C(\mathbf{T})$ is a normal unitary with parameters p, q, and

$$\varphi_{k_0,k_1} : M_n \otimes C(\mathbf{T}) \to M_n \otimes M_{k_0} \otimes C(\mathbf{T})$$

is a standard embedding with $k_1 \neq 0$. Then $\varphi_{k_0,k_1}(v)$ is normal in $M_n \otimes M_{k_0} \otimes C(\mathbf{T})$ with parameters $pk_1/l, qk_0/l$, where $l = \gcd(p, k_0)$. Hence if $k_0 \gg |p|$, then $\varphi_{k_0,k_1}(v)$ has small spectral variation.

We proceed to the construction of A using standard embeddings. By passing to a subsequence if necessary we may assume that $M(\chi^0_{n+1}) \geq \max(2L(\chi^1_n), 4)$, where

$$M(\chi) = \min_{i,j} \chi_{i,j} \text{ and } L(\chi) = \max_{i,j} |\chi_{i,j}|$$

for a matrix χ. Choose a preimage of the unit $[1_A] \in K_0(A)$ in $\mathbf{Z}^{k_1}$ and denote its i^{th} coordinate by $[1, i]$; set $[n+1, i] = \sum_j \chi^0_n(i,j)[n,j]$, $A_n = \bigoplus_{k=1}^{k_n} M_{[n,k]} \otimes C(\mathbf{T})$ and define $\varphi_n : A_n \to A_{n+1}$ to be the block diagonal sum of standard embeddings of type $(\chi^0_n(i,j), \chi^1_n(i,j))$ (that is, the partial embedding from the j^{th} summand of A_n to the i^{th} summand of A_{n+1} is of the above type). Since the partial embedding from a central summand in A_n to a central summand in A_m with $m > n$ is a sum of composites of standard embeddings, a central unitary is mapped to a sum of normal unitaries which by the above assumption must have uniformly small spectral variation; it decreases by a factor of two at least when embedded into the next level (we require $M(\chi^0_{n+1}) \geq 4$ to deal with the case $\chi^1_n(i,j) = 0$). Then $\lim A_n$ is a simple $A\mathbf{T}$ algebra of real rank zero (small spectral variation guarantees real rank zero — see [BBEK, 1.3]). Hence, by Elliott's classification theorem for simple real rank zero $A\mathbf{T}$ algebras (cf. [E, 7.1]), $A \cong \lim A_n$.

1.7. In Corollary 3.13 we establish a sharper version of Elliott's theorem: given two unital simple $A\mathbf{T}$ algebras of real rank zero, A and B, then an invertible element in $KK(A,B)$ which preserves positivity and the class of the unit lifts to an isomorphism.

1.8. The attempt to understand an invariant for approximately inner automorphisms of simple unital $A\mathbf{T}$ algebras of real rank zero introduced by Elliott and Rørdam (see [ER, 4.5]) stimulated our interest in the question addressed in the present work; their analysis of the invariant for the case of the Bunce-Deddens algebra (see [ER, 4.12 i]) was particularly helpful. We thank George Elliott for several valuable conversations; we also wish to thank the staff of the Fields Institute, where much of this work was done, for their hospitality.

1.9. Some notational conventions: as usual we let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{T}$ denote the natural numbers, the integers, the reals, and the complex numbers of unit modulus. For $m \in \mathbf{N}$, let $\{e_i\}$ denote the canonical basis in $\mathbf{Z}^m$; a homomorphism $\chi : \mathbf{Z}^n \rightarrow \mathbf{Z}^m$ will be confused with the $m \times n$ matrix $\chi_{i,j}$ (or $\chi(i,j)$) for which $\chi(e_j) = \sum_i \chi_{i,j} e_i$.

2. The extension class of an approximately inner automorphism

2.1. Let A be an $A\mathbf{T}$ algebra given as the limit of an inductive system of circle algebras $\{A_n, \varphi_{mn}\}$, where $A_n = \bigoplus_{k=1}^{k_n} A_{n,k}$, $A_{n,k} = M_{[n,k]} \otimes C(\mathbf{T})$ and $\varphi_{mn} : A_n \rightarrow A_m$ is a unital homomorphism for $m > n$; to simplify notation write φ_n in place of $\varphi_{n+1,n}$. Also, let $\overline{\varphi}_n$ denote the canonical homomorphism from A_n to A. Let $v_{n,k}$ be the unitary in A_n which restricts to the canonical central unitary in the k^{th} summand and the identity in the other summands. Let $\{e_{ij}^{n,k}\}$ denote the canonical family of matrix units for the k^{th} summand and let $p_{n,k}$ denote the projection onto the k^{th} summand (so $p_{n,k} = \sum_i e_{ii}^{n,k}$). Set $R_n = \{e_{ij}^{n,k} : 1 \leq i,j \leq [n,k], 1 \leq k \leq k_n\} \cup \{v_{n,k} : 1 \leq k \leq k_n\}$ and $S_n = R_n \cup \varphi_{n-1}(S_{n-1})$, where S_0 is the empty set.

2.2. We consider a family of approximately inner automorphisms and determine the resulting Ext classes in the following theorem. Let $\alpha_n = \operatorname{Ad} u_n$ be an inner automorphism of A_n, where u_n is a unitary in A_n; α_n and $\delta_n > 0$ are chosen inductively as follows: If φ and φ' are homomorphisms from A_n to A_{n+1} such that $||\varphi(x) - \varphi'(x)|| < \delta_n$ for $x \in R_n$, then $||\varphi(x) - \varphi'(x)|| < 2^{-n}$ for $x \in S_n$. Choose α_{n+1} such that

$$
\begin{aligned}
||\alpha_{n+1}^l \circ \varphi_n(e_{ij}^{n,k}) - \varphi_n \circ \alpha_n^l(e_{ij}^{n,k})|| &< \delta_n/32[n,k]k_n, \\
||\alpha_{n+1}^l \circ \varphi_n(v_{n,k}) - \varphi_n \circ \alpha_n^l(v_{n,k})|| &< \delta_n/32
\end{aligned}
\tag{1}
$$

for $l = \pm 1$. Since R_n generates A_n, the limit

$$
\lim_{m \longrightarrow \infty} \overline{\varphi}_m \circ \alpha_m \circ \varphi_{mn}(x) = \lim_{m \longrightarrow \infty} \operatorname{Ad} \overline{\varphi}_m(u_m)(\overline{\varphi}_n(x))
$$

exists for all $x \in R_n$. One defines a homomorphism $\alpha : \bigcup_{n=1}^{\infty} \overline{\varphi}_n(A_n) \rightarrow A$ by

$$
\alpha(\overline{\varphi}_n(x)) = \lim_{m \longrightarrow \infty} \overline{\varphi}_m \circ \alpha_m \circ \varphi_{mn}(x)
$$

for $x \in A_n$; note that α extends to a unital endomorphism. Since $\beta \circ \overline{\varphi}_n(x) = \lim \overline{\varphi}_m \circ {\alpha_m}^{-1} \circ \overline{\varphi}_{mn}(x)$ for $x \in A_n$ defines a unital endomorphism β of A and $\alpha \circ \beta \circ \overline{\varphi}_n(x) = \overline{\varphi}_n(x)$ for $x \in S_n$, it follows that α is an automorphism of A.

2.3. We view elements of A_n as matrix valued functions on k_n copies of $\mathbf{T}$ (the size of the matrix may vary). Since

$$||\alpha_{n+1} \circ \varphi_n(v_{n,j}) - \varphi_n(v_{n,j})|| < 2^{-n},$$

the Bott number $B(\varphi_n(v_{n,j})(\iota_i), u_{n+1}(\iota_i))$ (see 1.4) is well-defined (where ι_i is the basepoint of the i^{th} copy of $\mathbf{T}$ in the spectrum of A_{n+1}). For the purposes of computing the Bott number we may suppose that $u_{n+1}(\iota_i)$ commutes with $\varphi_n(p_{n,j})(\iota_i)$ (by perturbing it slightly) and so regard both $\varphi_n(v_{n,j})(\iota_i)$ and $u_{n+1}(\iota_i)$ as unitaries in $\varphi_n(p_{n,j})(\iota_i)M_{[n+1,i]}\varphi_n(p_{n,j})(\iota_i)$. Since the image of $A_{n,j}$ in

$$\varphi_n(p_{n,j})(\iota_i)M_{[n+1,i]}\varphi_n(p_{n,j})(\iota_i)$$

almost commutes with $\varphi_n(v_{n,j})(\iota_i)$ and $u_{n+1}\varphi_n(u_n^*)(\iota_i)$, it follows that

$$B(\varphi_n(v_{n,j})(\iota_i), u_{n+1}(\iota_i)) = B(\varphi_n(v_{n,j})(\iota_i), u_{n+1}(\iota_i)\varphi_n(u_n^*)(\iota_i))$$

is a multiple of $[n,j]$. For $i = 1, \dots, k_{n+1}$ and $j = 1, \dots, k_n$ set

$$\psi_n^0(i,j) = \begin{cases} B(\varphi_n(v_{n,j})(\iota_i), u_{n+1}(\iota_i))/[n,j] & \text{if } \varphi_n(p_{n,j})p_{n+1,i} \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

(note that this is an integer) and

$$\psi_n^1(i,j) = \begin{cases} [\varphi_n(p_{n,j})u_{n+1}p_{n+1,i}\,\varphi_n(u_n^*p_{n,j})]_1/[n,j] & \text{if } \varphi_n(p_{n,j})p_{n+1,i} \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

where $[\,\cdot\,]_1$ denotes the class of an invertible element in

$$K_1(\varphi_n(p_{n,j})A_{n+1,i}\varphi_n(p_{n,j})) \cong \mathbf{Z}.$$

Since $u_{n+1}p_{n+1,i}\varphi_n(u_n^*p_{n,j})$ almost commutes with the image of $A_{n,j} = M_{[n,j]}$, $\psi_n^1(i,j)$ is again an integer (for n sufficiently large).

$K_i(A)$ is given as the inductive limit of the sequence

$$\mathbf{Z}^{k_1} \xrightarrow{\chi_1^i} \mathbf{Z}^{k_2} \xrightarrow{\chi_2^i} \mathbf{Z}^{k_3} \to \cdots,$$

where $\chi_n^i = (\varphi_n)_*$, viewed as a $k_{n+1} \times k_n$ matrix with integer entries (note that $\chi_n^0(i,j) \geq 0$).

2.4. Theorem. *Let A, A_n, α, α_n, ψ_n^i, χ_n^i be as above. Then for $i = 0, 1$ the extension*

$$0 \to K_{i+1}(A) \to K_i(M_\alpha) \to K_i(A) \to 0$$

is obtained as the inductive limit of the sequence

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots\\
0 & \to & \mathbf{Z}^{k_n} & \xrightarrow{\iota} & \mathbf{Z}^{k_n} & \oplus & \mathbf{Z}^{k_n} & \xrightarrow{\rho} & \mathbf{Z}^{k_n} & \to & 0\\
 & & \chi_n^{i+1}\downarrow & & \chi_n^{i+1}\downarrow & \swarrow\psi_n^{i+1} & \downarrow\chi_n^i & & \downarrow\chi_n^i & & \\
0 & \to & \mathbf{Z}^{k_{n+1}} & \xrightarrow{\iota} & \mathbf{Z}^{k_{n+1}} & \oplus & \mathbf{Z}^{k_{n+1}} & \xrightarrow{\rho} & \mathbf{Z}^{k_{n+1}} & \to & 0\\
 & & \downarrow & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots
\end{array}$$

where ι is the embedding onto the first summand $\mathbf{Z}^{k_n}$ and ρ is the projection onto the second summand $\mathbf{Z}^{k_n}$ for each n; note that in each case K_0 is regarded as an ordered abelian group (while K_1 is regarded as an abelian group).

2.5. We proceed now to the proof of the theorem. For the inner automorphism α_n of A_n the six-term periodic sequence for the mapping torus M_{α_n} reduces to two trivial short exact sequences:

$$0 \rightarrow K_{i+1}(A_n) \rightarrow K_i(M_{\alpha_n}) \rightarrow K_i(A_n) \rightarrow 0$$

for $i = 0, 1$; note that $K_i(A_n)$ is naturally isomorphic to $\mathbf{Z}^{k_n}$ for $i = 0, 1$. The identification of $K_0(M_{\alpha_n})$ (resp. $K_1(M_{\alpha_n})$) with $\mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n}$ as ordered abelian groups (resp. abelian groups) proceeds as follows: Let $v : [0,1] \to U(M_2 \otimes A_n)$ be a continuous path of unitaries with

$$v_0 = \begin{pmatrix} u_n & 0 \\ 0 & u_n^* \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $x \oplus y \in M_{\alpha_n} \oplus M_{\alpha_n^{-1}}$ set

$$(x \oplus y)^{\sim}(t) = v_t \begin{pmatrix} x(t) & 0 \\ 0 & y(t) \end{pmatrix} v_t^*. \tag{2}$$

One checks that

$$(x \oplus y)^{\sim}(0) = (x \oplus y)^{\sim}(1);$$

thus $(x \oplus y)^{\sim}$ may be regarded as an element of $C(\mathbf{T}) \otimes M_2 \otimes A_n$. This defines an embedding of $M_{\alpha_n} \oplus M_{\alpha_n^{-1}}$ into $C(\mathbf{T}) \otimes M_2 \otimes A_n$. Since the image of the map $M_{\alpha_n} \to C(\mathbf{T}) \otimes M_2 \otimes A_n$ (by $x \mapsto (x \oplus 0)^{\sim}$) is a full hereditary subalgebra, we obtain an isomorphism,

$$K_*(M_{\alpha_n}) \simeq K_*(C(\mathbf{T}) \otimes M_2 \otimes A_n).$$

To identify $K_i(C(\mathbf{T}) \otimes M_2 \otimes A_n)$ with $\mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n}$ it suffices to identify $K_i(C(\mathbf{T}) \otimes M_2 \otimes A_{n,k})$ with $\mathbf{Z} \oplus \mathbf{Z}$ in the usual way — note that

$$C(\mathbf{T}) \otimes M_2 \otimes A_{n,k} = C(\mathbf{T}) \otimes M_2 \otimes M_{[n,k]} \otimes C(\mathbf{T}) \simeq M_{2[n,k]} \otimes C(\mathbf{T}^2).$$

The class of a unitary $u \in M_{2[n,k]} \otimes C(\mathbf{T}^2)$ is identified with $([u(\cdot, 1)]_1, [u(1, \cdot)]_1)$ (where the first variable is identified with the parameter in the mapping torus construction) and the class of a projection, $p \in M_{2[n,k]} \otimes C(\mathbf{T}^2)$, is identified with $(\mathrm{ch}(p), \dim(p))$, where $\mathrm{ch}(p)$ is the first Chern class of p and $\dim(p)$ is the dimension of p. Under this identification a nonzero element (m, n) is positive if and only if $n > 0$.

2.6. Lemma. *With identifications as above the extension*

$$0 \rightarrow K_{i+1}(A_n) \rightarrow K_i(M_{\alpha_n}) \rightarrow K_i(A_n) \rightarrow 0,$$

for $i = 0, 1$ is isomorphic to

$$0 \rightarrow \mathbf{Z}^{k_n} \xrightarrow{\iota} \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n} \xrightarrow{\rho} \mathbf{Z}^{k_n} \rightarrow 0$$

where $\iota(a) = (a, 0)$ and $\rho(a, b) = b$ for $a, b \in \mathbf{Z}^{k_n}$.

Proof. Let u be a unitary in A_n and let $w : [0,1] \to U(M_2 \otimes A_n)$ be a continuous path of unitaries with

$$w_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Set $e_t = w_t(1 \oplus 0)w_t^*$; then e is a projection in $M_2 \otimes (SA)^{\sim}$ and the map $K_1(A_n) \to K_0(M_{\alpha_n})$, is given by $[u] \mapsto [e] - [1]$ (see [B, 8.2.2]). By the above identification the second component of $[e] - [1]$ zero. Since the second component of $K_0(M_{\alpha_n})$

corresponds to dimension, the map $K_0(M_{\alpha_n}) \to K_0(A_n)$ is just the projection onto the second component.

Let p be a projection in A_n. Set $u_t = (e^{2\pi it} - 1)p + 1$. Then u is a unitary in $(SA)^\sim \subset M_{\alpha_n}$, and $[p]$ maps to $[u]$ under the map $K_0(A_n) \to K_1(M_{\alpha_n})$. Set $\tilde{u}(t) = v_t(u_t \oplus 1)v_t^*$, where $\{v_t\}$ is as above (see equation (2)); we now compute $[\tilde{u}]_1$ (in $K_1(C(\mathbf{T}) \otimes M_2 \otimes A_n)$). Since $\tilde{u}(1) = 1$, the second component of $[\tilde{u}]$ must be zero. The winding number of

$$t \mapsto \det \tilde{u}(t)(\iota_i) = \det((e^{2\pi it} - 1)p(\iota_i) + 1)$$

with $\tilde{u}(t)$ evaluated at the point ι_i in the spectrum of A_n is just $\dim p(\iota_i) = \dim(p \cdot p_{n,i})$. Thus the first component of $[\tilde{u}]$ is equal to $[p]$. Let u be a unitary in M_{α_n}, and set

$$\tilde{u}(t) = v_t(u_t \oplus 1)v_t^*.$$

Then, since $\tilde{u}(1) = u_1 \oplus 1$, $[u] = (*, [u_1])$ maps to $[u_1]$ under the map $K_1(M_{\alpha_n}) \to K_1(A_n)$. □

2.7. We now define a homomorphism from M_{α_n} to $M_{\alpha_{n+1}}$. This is done by using the fact that $\varphi_n \circ \alpha_n$ and $\alpha_{n+1} \circ \varphi_n$ are homotopic (as homomorphisms from A_n to A_{n+1}). The induced map, $K_i(M_{\alpha_n}) \to K_i(M_{\alpha_{n+1}})$ (for $i = 0, 1$), is computed in the following lemma.

Recall that we have chosen $\{\alpha_n\}$ so that

$$\begin{aligned} ||\alpha_{n+1} \circ \varphi_n(e_{ij}^{n,k}) - \varphi_n \circ \alpha_n(e_{ij}^{n,k})|| &< \delta_n/32[n,k]k_n, \\ ||\alpha_{n+1} \circ \varphi_n(v_{n,k}) - \varphi_n \circ \alpha_n(v_{n,k})|| &< \delta_n/32. \end{aligned}$$

We assume that $\delta_n > 0$ is sufficiently small in the following. Set

$$x = \sum_{i,k} \alpha_{n+1} \circ \varphi_n(e_{ii}^{n,k})\varphi_n \circ \alpha_n(e_{ii}^{n,k});$$

then $||x - 1|| < \delta_n/32$, and the unitary $v_1 = x|x|^{-1}$ in A_{n+1} satisfies

$$\mathrm{Ad}\, v_1 \circ \varphi_n \circ \alpha_n(e_{ii}^{n,k}) = \alpha_{n+1} \circ \varphi_n(e_{ii}^{n,k})$$

and $||v_1 - 1|| < \delta_n/16$ (for large n). It follows that for $i \neq j$

$$||\mathrm{Ad}\, v_1 \circ \varphi_n \circ \alpha_n(e_{ij}^{n,k}) - \alpha_{n+1} \circ \varphi_n(e_{ij}^{n,k})|| < 5\delta_n/32[n,k]k_n < 5\delta_n/32.$$

Now set

$$v_2 = \sum_{i,k} \mathrm{Ad}\, v_1 \circ \varphi_n \circ \alpha_n(e_{i1}^{n,k})\alpha_{n+1} \circ \varphi_n(e_{1i}^{n,k});$$

then v_2 is a unitary such that $||v_2 - 1|| \leq 5\delta_n/32$ and $\mathrm{Ad}\, v_2 v_1 \circ \varphi_n \circ \alpha_n(e_{ij}^{n,k}) = \alpha_{n+1} \circ \varphi_n(e_{ij}^{n,k})$. Since $||v_2 v_1 - 1|| < 7\delta_n/32$, there is a self-adjoint $h \in A_{n+1}$ such that $2\pi||h|| \leq \delta_n/4$ and $v_2 v_1 = e^{2\pi ih}$. Since

$$||\mathrm{Ad}\, e^{2\pi ih} \circ \varphi_n \circ \alpha_n(v_{n,k}) - \alpha_{n+1} \circ \varphi_n(v_{n,k})|| < 15\delta_n/32,$$

there is a self-adjoint $h_k \in A_{n+1}$ such that $2\pi||h_k|| < 16\delta_n/32$ and $e^{2\pi ih_k} = \alpha_{n+1} \circ \varphi_n(v_{n,k})\mathrm{Ad}\, e^{2\pi ih} \circ \varphi_n \circ \alpha_n(v_{n,k}^*)$. For $t \in [0,1]$ we define a homomorphism $\nu_t : A_n \to A_{n+1}$ as follows: for $0 \leq t \leq 1/2$, set

$$\nu_t = \mathrm{Ad}\, e^{4\pi ith} \circ \varphi_n \circ \alpha_n = \mathrm{Ad}\, (e^{4\pi ith}\varphi_n(u_n)) \circ \varphi_n,$$

and for $1/2 < t \le 1$, set

$$\begin{aligned}\nu_t(e_{ij}^{n,k}) &= \alpha_{n+1}\circ\varphi_n(e_{ij}^{n,k}),\\ \nu_t(v_{n,k}) &= e^{2\pi i(2t-1)h_k}\operatorname{Ad} e^{2\pi i h}\circ\varphi_n\circ\alpha_n(v_{n,k}).\end{aligned}$$

Note that ν_t is well defined, $t\mapsto\nu_t(x)$ is continuous for all $x\in A_n$, $\nu_0=\varphi_n\circ\alpha_n$ and $\nu_1=\alpha_{n+1}\circ\varphi_n$. Moreover, $||\nu_t(x)-\varphi_n\circ\alpha_n(x)||\le 2^{-n}$ for all $x\in S_n$ (since $||\nu_t(x)-\varphi_n\circ\alpha_n(x)||<\delta_n$ for $x\in R_n$). Define $\Phi_n: M_{\alpha_n}\to M_{\alpha_{n+1}}$ by

$$\Phi_n(x)(t)=\begin{cases}\varphi_n(x(\frac{t}{1-2^{-n}})), & 0\le t\le 1-2^{-n},\\ \nu_{2^n(t-1+2^{-n})}(x(0)), & 1-2^{-n}<t\le 1.\end{cases}$$

2.8. Lemma. *Let Φ_n be as above. Then, for $i=0,1$, $(\Phi_n)_*: K_i(M_{\alpha_n})\to K_i(M_{\alpha_{n+1}})$ is given by*

$$\begin{array}{ccc}\mathbf{Z}^{k_n} & \oplus & \mathbf{Z}^{k_n}\\ \chi_n^{i+1}\downarrow & \swarrow\psi_n^{i+1} & \downarrow\chi_n^i\\ \mathbf{Z}^{k_{n+1}} & \oplus & \mathbf{Z}^{k_{n+1}}\end{array}$$

Proof. We deal with the case $i=0$ first. Let $u\in A_n$ be a unitary and $\{w_t\}$ a continuous path of unitaries in $M_2\otimes A_n$ from $u\oplus u^*$ to 1. Set $e_t=w_t(1\oplus 0)w_t^*$. Since $e_0=1\oplus 0$, $\Phi_n(e)$ is equivalent to the projection defined by $t\mapsto\varphi_n(e_t)$, which is defined by $\varphi_n(u)$ in the same way as e is defined by u. Since $\Phi_n(1)=1$ and $[u]$ is mapped to $[e]-[1]$, this shows that χ_n^1 gives the map from the first summand of $K_0(M_{\alpha_n})$ to the first summand of $K_0(M_{\alpha_{n+1}})$ (a similar argument shows that χ_n^0 is the analogous map for $i=1$).

Let $p_{n,k}$ be the central projection in A_n as before. Regarding $p_{n,k}$ as a projection in M_{α_n}, we have

$$[p_{n,k}]=([u_np_{n,k}],[n,k])\in\mathbf{Z}\oplus\mathbf{Z}\subset\mathbf{Z}^{k_n}\oplus\mathbf{Z}^{k_n},$$

where $\mathbf{Z}\oplus\mathbf{Z}$ is mapped to the k^{th} summand of $(\mathbf{Z}\oplus\mathbf{Z})^{k_n}=\mathbf{Z}^{k_n}\oplus\mathbf{Z}^{k_n}$. Let $\{v_t\}$ be a continuous path of unitaries in $M_2\otimes A_{n+1}$ from $u_{n+1}\oplus u_{n+1}^*$ to 1. Since $u_{n+1}\oplus u_{n+1}^*$ almost commutes with $\varphi_n(p_{n,k})\oplus\varphi_n(p_{n,k})$, we can assume that v_t almost commutes with $\varphi_n(p_{n,k})\oplus\varphi_n(p_{n,k})$ for all $t\in[0,1]$. It follows that the class of $\Phi_n(p_{n,k})$ in $\mathbf{Z}^{k_{n+1}}\oplus\mathbf{Z}^{k_{n+1}}$ is the equivalence class of $t\mapsto v_t(\varphi_n(p_{n,k})\oplus 0)v_t^*$ composed with a short path from $\varphi_n(p_{n,k})\oplus 0$ to $\alpha_{n+1}\circ\varphi_n(p_{n,k})\oplus 0$. Since this corresponds to the invertible element $\varphi_n(p_{n,k})u_{n+1}\varphi_n(p_{n,k})+1-\varphi_n(p_{n,k})$, the first component of $[\Phi_n(p_{n,k})]$ is

$$\begin{aligned}[\varphi_n(p_{n,k})u_{n+1}\varphi_n(p_{n,k})]_1 &= [\varphi_n(p_{n,k})u_{n+1}\varphi_n(u_n^*)\varphi_n(p_{n,k})]_1+[\varphi_n(u_n)\varphi_n(p_{n,k})]_1\\ &= (\psi_n^1(i,k)[n,k]+\chi_n^1(i,k)[u_np_{n,k}])_i,\end{aligned}$$

where $[\,\cdot\,]_1$ denotes the K_1-class of an invertible element. The second component of $[\Phi_n(p_{n,k})]$ is the K_0-class of $\varphi_n(p_{n,k})$, i.e. $(\chi_n^0(i,k)[n,k])$. This takes care of the case $i=0$.

Regarding $v_{n,k}$ as an element of M_{α_n}, it follows that $[\Phi_n(v_{n,k})]$ is the class of

$$U: t\mapsto v_t(\Phi_n(v_{n,k})(t)\oplus 1)v_t^*$$

in $K_1(C(\mathbf{T})\otimes M_2\otimes A_{n+1})$. Evaluate U at ι_i (the base point of the i^{th} copy of $\mathbf{T}$ in the spectrum of A_{n+1}) and let w_i be the winding number of

$$\begin{aligned}t &\mapsto \det(v_t(\iota_i)(\Phi_n(v_{n,k})(t,\iota_i)\oplus 1)v_t(\iota_i)^*)\\ &= \det(\Phi_n(v_{n,k})(t,\iota_i)\oplus 1).\end{aligned}$$

By the definition of $\Phi_n(v_{n,k})$, w_i is equal to $\mathrm{Tr}(h_k(\iota_i))$, where h_k is defined by

$$e^{2\pi i h_k} = \alpha_{n+1} \circ \varphi_n(v_{n,k}) \mathrm{Ad}\, e^{2\pi i h} \circ \varphi_n \circ \alpha_n(v_{n,k}^*)$$

with h_k small. Since

$$e^{-2\pi i h} e^{2\pi i h_k} e^{2\pi i h} = e^{-2\pi i h} u_{n+1} \cdot \varphi_n(v_{n,k}) \cdot (e^{-2\pi i h} u_{n+1})^* \cdot \varphi_n(v_{n,k})^*,$$

if follows that w_i is equal to

$$B(\varphi_n(v_{n,k})(\iota_i), e^{-2\pi i h(\iota_i)} u_{n+1}(\iota_i)) = B(\varphi_n(v_{n,k})(\iota_i), u_{n+1}(\iota_i)) = \psi_n^0(i,k)[n,k].$$

Note that $[U(0)] = [\varphi_n(v_{n,k})]$; thus, $[\Phi_n(v_{n,k})] = (\psi_n^0(i,k)[n,k], \chi_n^1(i,k)[n,k])_i$. Since $[v_{n,k}] = (0,[n,k]) \in \mathbf{Z} \oplus \mathbf{Z} \subset \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n}$, the proof in the case $i = 1$ is complete. □

2.9. Lemma. *For any $x \in M_{\alpha_n}$, the limit*

$$\lim_{m \to \infty} \overline{\varphi}_{n+m} \circ \Phi_{n+m-1} \circ \cdots \circ \Phi_n(x)$$

exists in $C[0,1] \otimes A$ and defines a homomorphism of M_{α_n} to M_α.

Proof. Given $x \in M_{\alpha_n}$, then for all $t \in [0,1]$, $\overline{\varphi}_{n+m} \circ \Phi_{n+m-1} \circ \cdots \circ \Phi_n(x)(t)$ converges, say to $\tilde{x}(t)$; one has $\tilde{x}(1) = \alpha(\tilde{x}(0))$ and $\tilde{x}(0) = x(0)$. It also follows that convergence is uniform on $[0,t]$ for all $t < 1$. For $x \in S_n$ the family of functions, $\{\overline{\varphi}_{n+m} \circ \Phi_{n+m-1} \circ \cdots \circ \Phi_n(x)(t) : m > 0\}$ is uniformly continuous in a neighbourhood of $t = 1$. Hence convergence is uniform for all $x \in M_{\alpha_n}$. □

2.10. We conclude the proof of the theorem. By the previous lemma we have a homomorphism $\mu_n : M_{\alpha_n} \to M_\alpha$ such that $\mu_n = \mu_{n+1} \circ \Phi_n$. Let E denote the C*-inductive limit, $\lim_n M_{\alpha_n}$; there is a homomorphism $\mu : E \to M_\alpha$ such that $\mu_n = \mu \circ \overline{\Phi}_n$, where $\overline{\Phi}_n : M_{\alpha_n} \to E$ is the canonical map. We show that μ is an isomorphism.

If $\ker \mu \neq 0$, then $\ker \mu \cap \mathrm{im}\, \overline{\Phi}_n \neq 0$ for some n. Let $x \in M_{\alpha_n}$ be such that $\mu \circ \overline{\Phi}_n(x) = \mu_n(x) = 0$; then, since $\mu_n(x)(t) = \overline{\varphi}_n(x(t/s_n))$ for $0 \le t \le s_n$, where $s_n = \prod_n^\infty (1 - 2^{-k}) > 1 - 2^{-n+1}$, if follows that $\overline{\varphi}_n(x(t)) = 0$, and thus $\overline{\Phi}_n(x)(t) = 0$ for $0 \le t \le 1 - 2^{-n+1}$. Since $\overline{\Phi}_n(x) = \overline{\Phi}_m \circ \Phi_{mn}(x)$ for $m > n$, it follows that $\overline{\Phi}_n(x) = 0$; thus, μ is injective.

Let $x \in M_\alpha$ with $x(0) \in \overline{\varphi}_k(S_k)$ for some k and $\epsilon > 0$. There exist an n and $y \in C[0,1] \otimes A_n$ such that $||x(t) - \overline{\varphi}_n(y(t))|| < \epsilon$. Since

$$\begin{aligned} &||\overline{\varphi}_m \circ \alpha_m \circ \varphi_{mn}(y(0)) - \overline{\varphi}_n(y(1))|| \\ &\quad \le ||\overline{\varphi}_m \circ \alpha_m \circ \varphi_{mn}(y(0)) - \alpha \circ \overline{\varphi}_m \circ \varphi_{mn}(y(0))|| \\ &\quad\quad + ||\alpha \circ \overline{\varphi}_n(y(0)) - \alpha(x(0))|| + ||x(1) - \overline{\varphi}_n(y(1))|| \\ &\quad \le \epsilon_m + 2\epsilon, \end{aligned}$$

where $\epsilon_m \to 0$ as $m \to \infty$, we may take a small modification of y around $t = 1$ and obtain, for sufficiently large $m > n$, $y \in C[0,1] \otimes A_m$ such $y(0) \in S_n$, $\overline{\varphi}_m(y(0)) = x(0)$, $y \in M_{\alpha_m}$, and $||x(t) - \overline{\varphi}_m(y(t))|| < 3\epsilon$. If m is sufficiently large we have that $||\overline{\varphi}_m(y(t)) - \mu_m(y)(t)|| < 7\epsilon$, since $\mu_m(y)$ is obtained by extending $t \mapsto \overline{\varphi}_m(y(t))$ beyond $t = 1$ up to $1/s_m$ with s_m defined above and rescaling it; the extended part is within a sphere of radius 2^{-m+1} centered at $\overline{\varphi}_m(y(1))$. Thus we have that $||x - \mu_m(y)|| < 11\epsilon$. This implies that μ is surjective.

3. Realization of an extension class by an approximately inner automorphism

3.1. The proof of the following theorem will be divided into two cases; the case $i = 1$ will be dealt with in sections 3.8 and 3.9 and the case $i = 0$ will be handled in 3.11.

Theorem. *Let A be a simple unital $A\mathbf{T}$ algebra of real rank zero, and let $i = 0$ or 1. Then, given an extension*

$$0 \rightarrow K_{i+1}(A) \rightarrow E \rightarrow K_i(A) \rightarrow 0,$$

there is an approximately inner automorphism α of A such that

$$0 \rightarrow K_{i+1}(A) \rightarrow K_i(M_\alpha) \rightarrow K_i(A) \rightarrow 0$$

is equivalent to the above extension and

$$0 \rightarrow K_i(A) \rightarrow K_{i+1}(M_\alpha) \rightarrow K_{i+1}(A) \rightarrow 0$$

is a trivial extension. That is, for every element $\eta \in \mathrm{Ext}(K_i(A), K_{i+1}(A))$ there is an approximately inner automorphism α of A such that $\eta_i(\alpha) = \eta$ and $\eta_{i+1}(\alpha) = 0$.

3.2. Lemma. *Given $\chi \in M_{mn}(\mathbf{Z})$ and a subgroup W of $\mathbf{Z}^m$ with $\mathrm{im}\,\chi \cap W = \{0\}$, there is a constant $c \geq 0$ such that for any $\psi \in M_{mn}(\mathbf{Z})$ with $\ker\psi \supset \ker\chi$ there is a $\gamma \in M_n(\mathbf{Z})$ such that*

$$\ker\gamma \supset W \quad \textit{and} \quad |(\psi - \gamma\chi)(i,j)| \leq c$$

for $i = 1, \ldots, m,\ j = 1, \ldots, n$.

Proof. Let $r = \mathrm{rank}\,\chi$ and let $b_1, \ldots, b_n$ be generators for $\mathbf{Z}^n$ such that $\chi b_1, \ldots, \chi b_r$ generate $\chi\mathbf{Z}^n$ and $\chi b_i = 0$ for $i = r+1, \ldots, n$. We denote by U the invertible matrix in $M_n(\mathbf{Z})$ defined by $(b_1 \cdots b_n)$.

Let $\tilde{W}$ be a maximal subgroup of $\mathbf{Z}^m$ such that

$$\tilde{W} \cap \mathrm{im}\,\chi = \{0\},\ \tilde{W} \supset W.$$

Then $\tilde{W}$ is isomorphic to $\mathbf{Z}^{m-r}$. Let $a_i = \chi b_i$ for $i = 1, \ldots, r$, and let $a_{r+1}, \ldots, a_m$ be generators for $\tilde{W}$. Define a matrix $V \in M_m(\mathbf{Z})$ by $V = (a_1 \cdots a_m)$. Note that the determinant $\det V$ is non-zero and that $|\det V| > 1$ if $\mathrm{im}\,\chi + \tilde{W} \neq \mathbf{Z}^m$. Then χ can be expressed by

$$\chi = V\tilde{E}_r U^{-1},$$

where $\tilde{E}_r$ is the canonical rank r matrix in $M_{mn}(\mathbf{Z})$ with 1 on the first r diagonal elements and 0 elsewhere. Since $\ker\psi \supset \ker\chi$, ψ can be expressed as

$$\begin{aligned}\psi &= (\psi(b_1) \cdots \psi(b_r)\, 0 \cdots 0)\tilde{E}_r U^{-1} \\ &= (\psi(b_1) \cdots \psi(b_r)\, 0 \cdots 0)V^{-1}\chi.\end{aligned}$$

Note that for any $\eta_1, \ldots, \eta_r$ in $\mathbf{Z}^m$ the matrix

$$\gamma = (\eta_1 \cdots \eta_r\, 0 \cdots 0)(\det V)V^{-1}$$

is such that $\gamma \in M_m(\mathbf{Z})$ and $\ker\gamma \supset \tilde{W} \supset W$. We choose $\eta_i \in \mathbf{Z}^m$ so that

$$\|(\det V)^{-1}\psi(b_i) - \eta_i\|_\infty \leq 1/2,$$

where $\|x\|_\infty = \max|x_j|$ for $x \in \mathbf{R}^m$. Then defining γ as above, we obtain that

$$\psi - \gamma\chi = \{(\det V)^{-1}(\psi(b_1) \cdots \psi(b_r)\, 0 \cdots) - (\eta_1 \cdots \eta_r\, 0 \cdots 0)\}(\det V)V^{-1}\chi$$

and

$$|(\psi - \gamma\chi)(i,j)| \le c,$$

where

$$c = \max_j 1/2 \sum_{k,l} |\det V| \cdot |V^{-1}(k,l)\chi(l,j)|,$$

which does not depend on ψ. □

3.3. Let G_1 be a countable torsion-free abelian group. Then there exist a sequence l_n of positive integers and an inductive system

$$\mathbf{Z}^{l_1} \xrightarrow{\chi_1^1} \mathbf{Z}^{l_2} \xrightarrow{\chi_2^1} \mathbf{Z}^{l_3} \to \cdots,$$

for which the inductive limit is isomorphic to G_1, such that

$$\operatorname{rank} \chi_{n+1}^1 \chi_n^1 = \operatorname{rank} \chi_n^1$$

for $n = 1, 2, 3, \ldots$. Since $\operatorname{rank} \chi_{n+1,m}^1 = \operatorname{rank} \chi_{nm}^1$ for $n > m$, we have that $\ker \chi_n^1 \cap \operatorname{im} \chi_{nm}^1 = \{0\}$ for $n > m$. Thus we obtain a constant $c_{nm} = c$ for the pair χ_{nm}^1 and $\ker \chi_n^1$ by the previous lemma. We set

$$c_n = \max\{c_{km} : n \ge k > m\},$$

which forms an increasing sequence. We always assume that $c_n \ge 1$.

Later, for a sequence k_n of positive integers with $k_n \ge l_n$, we define another inductive system

$$\mathbf{Z}^{k_1} \xrightarrow{\tilde{\chi}_1^1} \mathbf{Z}^{k_2} \xrightarrow{\tilde{\chi}_2^1} \mathbf{Z}^{k_3} \to \cdots,$$

by extending χ_n^1 with 0's, i.e., defining $\tilde{\chi}_n^1$ by

$$\mathbf{Z}^{k_n} \xrightarrow{\rho} \mathbf{Z}^{l_n} \xrightarrow{\chi_n^1} \mathbf{Z}^{l_{n+1}} \xrightarrow{\iota} \mathbf{Z}^{k_{n+1}},$$

where ρ (resp. ι) is an obvious projection (resp. embedding). The inductive limit is the same as before. Since $\ker \tilde{\chi}_{n+1}^1 = \iota(\ker \chi_{n+1}^1) \oplus \mathbf{Z}^{k_{n+1}-l_{n+1}}$ and $\operatorname{im} \tilde{\chi}_n^1 \subset \mathbf{Z}^{l_n} \oplus 0$, we retain the same constant c_n as before for this new inductive system.

Let G_0 be a simple dimension group other than $\mathbf{Z}$. Then there exist an increasing sequence m_n of positive integers and an inductive system

$$\mathbf{Z}^{m_1} \xrightarrow{\chi_1^0} \mathbf{Z}^{m_2} \xrightarrow{\chi_2^0} \mathbf{Z}^{m_3} \to \cdots,$$

for which the inductive limit is isomorphic to G_0 (as ordered abelian groups), such that

$$m_n \ge l_n, \; M(\chi_n^0) \ge n c_n,$$

where $M(\chi) = \min_{i,j} \chi(i,j)$ for a matrix χ with non-negative entries. The extra conditions placed on the inductive system can be handled easily.

3.4. Lemma. *Given an extension*

$$0 \to K_0(A) \xrightarrow{\iota} E \xrightarrow{q} K_1(A) \to 0$$

as in the theorem, there exist an increasing sequence k_n of positive integers and inductive systems

$$\mathbf{Z}^{k_1} \xrightarrow{\chi_1^i} \mathbf{Z}^{k_2} \xrightarrow{\chi_2^i} \mathbf{Z}^{k_3} \to \cdots$$

for $i = 0, 1$ and

$$\mathbf{Z}^{k_1} \oplus \mathbf{Z}^{k_1} \xrightarrow{\psi_1} \mathbf{Z}^{k_2} \oplus \mathbf{Z}^{k_2} \xrightarrow{\psi_2} \mathbf{Z}^{k_3} \oplus \mathbf{Z}^{k_3} \to \cdots$$

such that $M(\chi_n^0) \geq nc_n$ *with* c_n *defined for the sequence* χ_n^1 *as in 3.3,* ψ_n *is of the form* $\psi_n(a,b) = (\chi_n^0(a) + \psi_n^0(b), \chi_n^1(b))$*,* $\ker \psi_n^0 \supset \ker \chi_n^1$*,* $\operatorname{rank} \chi_{n+1}^1 \chi_n^1 = \operatorname{rank} \chi_n^1$*, and the given short exact sequence is isomorphic to the inductive limit of the sequence*

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}^{k_1} & \xrightarrow{\iota} & \mathbf{Z}^{k_1} \oplus \mathbf{Z}^{k_1} & \xrightarrow{\rho} & \mathbf{Z}^{k_1} & \to & 0 \\ & & \chi_1^0 \downarrow & & \psi_1 \downarrow & & \downarrow \chi_1^1 & & \\ 0 & \to & \mathbf{Z}^{k_2} & \xrightarrow{\iota} & \mathbf{Z}^{k_2} \oplus \mathbf{Z}^{k_2} & \xrightarrow{\rho} & \mathbf{Z}^{k_2} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

We may also assume that $M(\chi_{n+1}^0) \geq \max(2L(\chi_n^1), 4)$ *for all* n*.*

Proof. For $G_i = K_i(A)$ we choose inductive systems as in 3.3, $\mathbf{Z}^{m_1} \to \mathbf{Z}^{m_2} \to \cdots$ for $i = 0$ and $\mathbf{Z}^{l_1} \to \mathbf{Z}^{l_2} \to \cdots$ for $i = 1$, with all the properties specified there. We shall choose a subsequence $k(n)$ of positive integers and homomorphisms $\zeta_n : \mathbf{Z}^{m_{k(n)}} \oplus \mathbf{Z}^{l_n} \to E$, $\psi_n : \mathbf{Z}^{m_{k(n)}} \oplus \mathbf{Z}^{l_n} \to \mathbf{Z}^{m_{k(n+1)}} \oplus \mathbf{Z}^{l_{n+1}}$ for each n such that the diagrams

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}^{m_{k(n)}} & \xrightarrow{\iota} & \mathbf{Z}^{m_{k(n)}} \oplus \mathbf{Z}^{l_n} & \xrightarrow{\rho} & \mathbf{Z}^{l_n} & \to & 0 \\ & & \overline{\chi}_{k(n)}^0 \downarrow & & \zeta_n \downarrow & & \downarrow \overline{\chi}_n^1 & & \\ 0 & \to & K_0(A) & \longrightarrow & E & \longrightarrow & K_1(A) & \to & 0 \end{array}$$

and

$$\begin{array}{ccc} \mathbf{Z}^{m_{k(n)}} \oplus \mathbf{Z}^{l_n} & \xrightarrow{\psi_n} & \mathbf{Z}^{m_{k(n+1)}} \oplus \mathbf{Z}^{l_{n+1}} \\ \zeta_n \downarrow & & \downarrow \zeta_{n+1} \\ E & = & E \end{array}$$

are commutative, $M(\chi_{k(n+1)k(n)}^0) \geq nc_n$ (and $M(\chi_{k(n+1)k(n)}^0) \geq \max(2L(\chi_{n-1}^1), 4)$, for $n > 1$) and

$$\ker \psi_n^0 \supset \ker \chi_n^1, \quad \ker \zeta_n \supset \ker(\overline{\chi}_{k(n)}^0 \oplus \overline{\chi}_n^1),$$

where $\psi_n(a,b) = (\chi_{k(n+1)k(n)}^0(a) + \psi_n^0(b), \chi_n^1(b))$. Once this is done, we set $k_n = m_{k(n)}$ $(\geq l_n)$ and replace the inductive system $\mathbf{Z}^{l_1} \to \mathbf{Z}^{l_2} \to \cdots$ for $K_1(A)$ by

$$\mathbf{Z}^{k_1} \xrightarrow{\chi_1^1} \mathbf{Z}^{k_2} \to \cdots$$

as specified in 3.3. We also redefine ζ_n, ψ_n in the obvious way. The required properties are retained under this change, and thus the proof will be complete.

We prove the above assertion by induction. Let $k(1) = 1$. Since $\operatorname{im} \overline{\chi}_1^1$ is torsion-free, it is isomorphic to $\mathbf{Z}^r$ for some $r \leq l_1$. Hence we can find a homomorphism ϕ of $\operatorname{im} \overline{\chi}_1^1$ into E such that $q \circ \phi = \mathrm{id}$ and define ζ_1 by

$$\zeta_1(a,b) = \iota \circ \overline{\chi}_1^0(a) + \phi \circ \overline{\chi}_1^1(b).$$

Then the required properties for ζ_1 are immediate. Suppose that we have constructed $k(1), \ldots, k(n)$, $\zeta_1, \ldots, \zeta_n$, and $\psi_1, \ldots, \psi_{n-1}$ with the required properties. We then find a homomorphism $\phi : \operatorname{im} \overline{\chi}_{n+1}^1 \to E$ such that $q \circ \phi = \mathrm{id}$. Choose a basis $b_1, \ldots, b_{l_n} \in \mathbf{Z}^{l_n}$ so that $\overline{\chi}_n^1(b_1), \ldots, \overline{\chi}_n^1(b_r)$ generates $\operatorname{im} \overline{\chi}_n^1 \cong \mathbf{Z}^r$ and $\overline{\chi}_n^1(a_i) = 0$ for $r < i \leq l_n$. Then it follows that

$$\zeta_n(0, b_i) - \phi \circ \overline{\chi}_n^1(b_i) \in \iota(K_0(A)).$$

for all $1 \leq i \leq l_n$ (note that $\zeta_n(0,b_i) = 0 = \phi \circ \overline{\chi}_n^1(b_i)$ for $i > r$). Since $\bigcup_{n=1}^{\infty} \operatorname{im} \overline{\chi}_n^0 = K_0(A)$, there is a $k(n+1) > k(n)$ such that the left hand side is contained in

$\iota(\operatorname{im}\overline{\chi}^0_{k(n+1)})$ for all i — we also require that $M(\chi^0_{k(n+1)k(n)}) \geq \max(2L(\chi^1_{n-1}), 4)$ for $n > 1$.

We define $\zeta_{n+1} : \mathbf{Z}^{m_{k(n+1)}} \oplus \mathbf{Z}^{l_{n+1}} \to E$ by

$$\zeta_{n+1}(a,b) = \iota \circ \overline{\chi}^0_{k(n+1)}(a) + \phi \circ \overline{\chi}^1_{n+1}(b);$$

it follows by construction that $\ker \zeta_{n+1} \supset \ker(\overline{\chi}^0_{k(n+1)} \oplus \overline{\chi}^1_{n+1})$. Then there exist $p_i \in \mathbf{Z}^{m_{k(n+1)}}$ such that

$$\zeta_n(0,b_i) - \zeta_{n+1}(0,\chi^1_n(b_i)) = \zeta_n(0,b_i) - \phi \circ \overline{\chi}^1_n(b_i) = \iota \circ \overline{\chi}^0_{k(n+1)}(p_i).$$

We assume that $p_i = 0$ for $i > r$ and define $\psi^0_{n+1} : \mathbf{Z}^{l_n} \to \mathbf{Z}^{m_{k(n+1)}}$ by

$$\psi^0_{n+1}(b_i) = p_i.$$

Since the b_i's form a basis, ψ^0_{n+1} is well-defined; moreover, $\ker \psi^0_{n+1} \supset \ker \overline{\chi}^1_{n+1} = \ker \chi^1_{n+1}$, where the last equality follows from the assumption $\operatorname{rank} \chi^1_{n+1}\chi^1_n = \operatorname{rank} \chi^1_n$. Since $k(n+1) \geq n+1$ and hence $c_{k(n+1)} \geq c_{n+1}$ (from the definition of c_n), it follows that

$$M(\chi^0_{k(n+1)}) \geq k(n+1)c_{k(n+1)} \geq (n+1)c_{n+1},$$

where the first inequality follows from the choice made at the beginning of the proof. Since the commutativity of the diagrams follows immediately, this completes the induction. □

3.5. By passing to a subsequence one may further assume that $M(\chi^0_n) > L_n c_n$ for arbitrarily large L_n. First note that for $n > m$,

$$\psi_{nm}(a,b) = \psi_{n-1} \circ \cdots \circ \psi_m(a,b) = (\chi^0_{nm}(a) + \psi^0_{nm}(b), \chi^1_{nm}(b)),$$

where $\psi^0_{nm} : \mathbf{Z}^{k_m} \to \mathbf{Z}^{k_{n+1}}$ is defined by

$$\psi^0_{nm}(b) = \sum_{k=m}^{n-1} \chi^0_{n,k+1} \circ \psi_k \circ \chi^1_{km}(b).$$

Thus it follows that $\ker \psi^0_{nm} \supset \ker \overline{\chi}^1_m = \ker \chi^1_m$. In this way we show that by passing to a subsequence n_j in the situation of 3.4 all the algebraic conditions are retained and the estimate on $M(\chi^0_n)$ is improved as follows:

$$M(\chi^0_{n_{j+1}n_j}) \geq M(\chi^0_{n_{j+1}-1}) \geq (n_{j+1}-1)c_{n_{j+1}-1},$$

where $(c_{n_{j+1}-1})$ can be regarded as (c_j) for the sequence $(\chi^1_{n_{j+1}n_j})$ and $n_{j+1}-1$ can be made arbitrarily large.

3.6. Lemma. *Suppose that the extension*

$$0 \to K_0(A) \xrightarrow{\iota} E \xrightarrow{q} K_1(A) \to 0$$

is obtained as the inductive limit of

$$\begin{array}{ccccccccc}
0 & \to & \mathbf{Z}^{k_1} & \xrightarrow{\iota} & \mathbf{Z}^{k_1}\oplus\mathbf{Z}^{k_1} & \xrightarrow{\rho} & \mathbf{Z}^{k_1} & \to & 0 \\
 & & \chi^0_1\downarrow & & \psi_1\downarrow & & \downarrow\chi^1_1 & & \\
0 & \to & \mathbf{Z}^{k_2} & \xrightarrow{\iota} & \mathbf{Z}^{k_2}\oplus\mathbf{Z}^{k_2} & \xrightarrow{\rho} & \mathbf{Z}^{k_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where ψ_n is of the form $\psi_n(a,b) = (\chi_n^0(a) + \psi_n^0(b), \chi_n^1(b))$. Let $\gamma_n \in M_{k_n}(\mathbf{Z})$ be given for all $n \geq 1$ and define a homomorphism $\varphi_n : \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n} \to \mathbf{Z}^{k_{n+1}} \oplus \mathbf{Z}^{k_{n+1}}$ in the same way as ψ_n with ψ_n^0 replaced by

$$\varphi_n^0 = \psi_n^0 + \chi_n^0 \gamma_n - \gamma_{n+1} \chi_n^1.$$

Then the extension obtained as the inductive limit of the above system with φ_n in place of ψ_n is isomorphic to the original extension.

Proof. Define a homomorphism $\nu_n : \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n} \to \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n}$ by

$$\nu_n(a,b) = (a + \gamma_n(b), b).$$

Then $\nu_{n+1} \circ \varphi_n = \psi_n \circ \nu_n$ by computation. Thus (ν_n) induces an isomorphism of the inductive limit E' for (φ_n) onto E as required. One can show this by direct computation, or it also follows from the following commutative diagram by knowing that both rows are exact:

$$\begin{array}{ccccccccc} 0 & \to & K_0(A) & \to & E' & \to & K_1(A) & \to & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \to & K_0(A) & \to & E & \to & K_1(A) & \to & 0 \end{array}$$

□

3.7. Lemma. *Let φ be the standard n-times around embedding of $C(\mathbf{T})$ into $M_n \otimes C(\mathbf{T})$ and let u be the canonical unitary of $C(\mathbf{T})$. Then for any $k \in \{0, 1, \cdots, n-1\}$ there is a unitary $v \in M_n \otimes C(\mathbf{T})$ with $[v] = 0$ such that $v\varphi(u)v^* = e^{2\pi i k/n}\varphi(u)$.*

Proof. Recall that φ has the form

$$\varphi(u)(z) = \begin{pmatrix} 0 & 0 & \cdots & z \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix},$$

and define

$$v = \begin{pmatrix} \omega^0 & 0 & \cdots & 0 \\ 0 & \omega^1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix},$$

where $\omega = e^{2\pi i/n}$. Then $\operatorname{Ad} v(\varphi(u)) = \omega\varphi(u)$. □

3.8. *Proof of the theorem for the case $i = 1$.* By Lemma 3.4 we may express the short exact sequence.

$$0 \to K_0(A) \xrightarrow{\iota} E \xrightarrow{q} K_1(A) \to 0$$

as the limit of an inductive system of short exact sequences:

$$\begin{array}{ccccccccccc} 0 & \to & \mathbf{Z}^{k_1} & \xrightarrow{\iota} & \mathbf{Z}^{k_1} & \oplus & \mathbf{Z}^{k_1} & \xrightarrow{\rho} & \mathbf{Z}^{k_1} & \to & 0 \\ & & \chi_1^0\downarrow & & \chi_1^0\downarrow & \swarrow\psi_1^0 & \downarrow\chi_1^1 & & \downarrow\chi_1^1 & & \\ 0 & \to & \mathbf{Z}^{k_2} & \xrightarrow{\iota} & \mathbf{Z}^{k_2} & \oplus & \mathbf{Z}^{k_2} & \xrightarrow{\rho} & \mathbf{Z}^{k_2} & \to & 0 \\ & & \downarrow & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\ \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots \end{array}$$

where $M(\chi_n^0) > L_n c_n$ for all n and some constant $L_n \geq n$, $\ker \psi_n^0 \supset \ker \chi_n^1$, $\operatorname{rank} \chi_{n+1}^1 \chi_n^1 = \operatorname{rank} \chi_n^1$ and $M(\chi_{n+1}^0) \geq \max(2L(\chi_n^1), 4)$ for all n. Note by 3.5 that these properties hold with arbitrarily large L_n when passing to a subsequence.

We choose a subsequence of the above inductive system, replace ψ_n^0 so that $|\psi_n^0(i,j)| \leq c_n$ holds (Lemma 3.2) without changing the inductive limit (Lemma 3.6), and construct circle algebras $A_n = \bigoplus_{k=1}^{k_n} M_{[n,k]} \otimes C(\mathbf{T})$ for each level of the subsequence, embeddings $\varphi_n : A_n \to A_{n+1}$ by using standard embeddings (see 1.6), and unitaries $u_n \in A_n$ such that φ_n induces $\chi_n^i : K_i(A_n) \to K_i(A_{n+1})$, $\operatorname{Ad} u_{n+1} \circ \varphi_n \approx \varphi_n \circ \operatorname{Ad} u_n$ as described in the first theorem (which depends on how small $|\psi_n^0(i,j)|/M(\chi_n^0) \leq L_n^{-1}$ is), $[u_{n+1}, \varphi_n(p_{n,j})] = 0$,

$$[u_{n+1} p_{n+1,i} \varphi_n(u_n^* p_{n,j})]_1 = 0$$

in $K_1(\varphi_n(p_{n,j}) A_{n+1,i} \varphi_n(p_{n,j}))$ and

$$B(\varphi_n(v_{n,j})(\iota_i), u_{n+1}(\iota_i)) = \psi_n^0(i,j)[n,j].$$

Let A_∞ denote the inductive limit of $A_1 \to A_2 \to A_3 \to \dots$; then, by Elliott's classification theorem (cf. [E, 7.1]), A_∞ is isomorphic to A (by 1.6 the condition $M(\chi_{n+1}^0) \geq \max(2L(\chi_n^1), 4)$ for all n guarantees that A_∞ has real rank zero), and the limit $\alpha = \lim \operatorname{Ad} \overline{\varphi}_n(u_n)$ defines the desired automorphism α of A_∞. Note that $\eta_0(\alpha) = 0$ by the above properties of $[u_n]$.

Let u_1 be an arbitrary unitary in A_1 with $[u_1] = 0$. The construction now proceeds inductively; suppose that $A_1, \dots, A_n$, $\varphi_1, \dots, \varphi_{n-1}$, $\delta_1, \dots, \delta_{n-1}$, and $u_1, \dots, u_n$ have been chosen so that condition (1) on $\alpha_{m+1} \circ \varphi_m \approx \varphi_m \circ \alpha_m$ holds with $\alpha_m = \operatorname{Ad} u_m$. Recall that $S_{m+1} = R_{m+1} \cup \varphi_m(S_m)$ and R_m are the generators for A_m. □

3.9. We note the following lemma, without giving its proof.

Lemma. *Let F be a finite subset of A_n. For any $\epsilon > 0$ there is a $\delta > 0$ such that for any two homomorphisms φ, φ' from A_n into another C^*-algebra B, if $||\varphi(x) - \varphi'(x)|| < \delta$ for all $x \in R_n$, then $||\varphi(x) - \varphi'(x)|| < \epsilon$ for $x \in F$.*

By applying the above lemma for $F = S_n$ and $\epsilon = 2^{-n}$ we obtain $\delta = \delta_n > 0$. Then, by passing to a subsequence (see 3.5) and modifying ψ_n^0, ψ_{n+1}^0 by using 3.2 and 3.6, we may assume that

$$2\pi(L(\psi_n^0) + 1)/M(\chi_n^0) < \delta_n/32,$$

where $L(\psi) = \max_{i,j} |\psi(i,j)|$. Note that the modified ψ_{n+1}^0 satisfies the condition $\ker \psi_{n+1}^0 \supset \ker \chi_{n+1}^1$. Define an embedding $\varphi_n : A_n \to A_{n+1}$ so that the partial embedding from $A_{n,j}$ to $A_{n+1,i}$ is a standard embedding of type $(\chi_n^0(i,j), \chi_n^1(i,j))$ (see 1.6). Using 3.7 choose a unitary

$$v_{n+1} \in \varphi_n(\bigoplus_{j=1}^{k_n} M_{[1,j]} \otimes 1)' \cap \bigoplus_{i=1}^{k_{n+1}} M_{[n+1,i]} \otimes 1 \subset A_{n+1}$$

such that

$$\operatorname{Ad} v_{n+1} \circ \varphi_n(v_{nj}) p_{n+1,i} = e^{2\pi i \psi_n^0(i,j)/\chi_n^0(i,j)} \varphi_n(v_{nj}) p_{n+1,i}$$

in the case $\chi_n^1(i,j) \neq 0$, and

$$\operatorname{Ad} v_{n+1} \circ \varphi_n(v_{nj}) p_{n+1,i} \cong e^{2\pi i a_1/b_1} \varphi_{b_1,1}(v_{nj}) \oplus e^{2\pi i a_2/b_2} \varphi_{b_2,-1}(v_{nj})$$

in the case $\chi^1_n(i,j)=0$, where $\varphi_n(v_{nj})\cong\varphi_{b_1,1}(v_{nj})\oplus\varphi_{b_2,-1}(v_{nj})$, $a_1=\lfloor\psi^0_n(i,j)/2\rfloor$, $b_1=\lfloor\chi^0_n(i,j)/2\rfloor$, $a_1+a_2=\psi^0_n(i,j)$, and $b_1+b_2=\chi^0_n(i,j)$ (see 1.6). In either case it follows that $[v_{n+1}\varphi_n(p_{n,j})]_1=0$ in $K_1(\varphi_n(p_{n,j})A_{n+1}\varphi_n(p_{n,j}))$, and

$$\begin{aligned}&B(\varphi_n(v_{nj})(\iota_i),\,u_{n+1}(\iota_i))=\psi^0_n(i,j)[n,j],\\&||\mathrm{Ad}\,v_{n+1}\circ\varphi_n(v_{n,j})-\varphi_n(v_{n,j})||<\delta_n/32,\\&\mathrm{Ad}\,v_{n+1}\circ\varphi_n(e^{n,k}_{ij})=\varphi_n(e^{n,k}_{ij}).\end{aligned}$$

Setting $u_{n+1}=v_{n+1}\varphi_n(u_n)$ completes the induction.

3.10. Lemma. *Given an extension as in the theorem,*

$$0\;\to\;K_1(A)\;\xrightarrow{\iota}\;E\;\xrightarrow{q}\;K_0(A)\;\to\;0,$$

there exists an increasing sequence k_n of positive integers and inductive systems

$$\mathbf{Z}^{k_1}\;\xrightarrow{\chi^i_1}\;\mathbf{Z}^{k_2}\;\xrightarrow{\chi^i_2}\;\mathbf{Z}^{k_3}\;\to\;\cdots$$

for $i=0,1$ and

$$\mathbf{Z}^{k_1}\oplus\mathbf{Z}^{k_1}\;\xrightarrow{\psi_1}\;\mathbf{Z}^{k_2}\oplus\mathbf{Z}^{k_2}\;\xrightarrow{\psi_2}\;\mathbf{Z}^{k_3}\oplus\mathbf{Z}^{k_3}\;\to\;\cdots$$

where χ^0_n is positive, and ψ_n is of the form $\psi_n(a,b)=(\chi^1_n(a)+\psi^1_n(b),\chi^0_n(b))$ for each n, such that the given short exact sequence is isomorphic to the inductive limit of the sequence

$$\begin{array}{ccccccccc}0&\to&\mathbf{Z}^{k_1}&\xrightarrow{\iota}&\mathbf{Z}^{k_1}\oplus\mathbf{Z}^{k_1}&\xrightarrow{\rho}&\mathbf{Z}^{k_1}&\to&0\\&&\chi^1_1\downarrow&&\psi_1\downarrow&&\downarrow\chi^0_1&&\\0&\to&\mathbf{Z}^{k_2}&\xrightarrow{\iota}&\mathbf{Z}^{k_2}\oplus\mathbf{Z}^{k_2}&\xrightarrow{\rho}&\mathbf{Z}^{k_2}&\to&0\\&&\downarrow&&\downarrow&&\downarrow&&\\\vdots&&\vdots&&\vdots&&\vdots&&\vdots\end{array}$$

Before going into the proof, note that E has a natural order, $a\in E$ is positive if $q(a)$ is a non-zero positive element of $K_0(A)$ or $a=0$, and that E is a dimension group with respect to this order. Define an order on $\mathbf{Z}^{k_n}\oplus\mathbf{Z}^{k_n}$ by $(a,b)\ge0$ if $b\ge0,\ b\ne0$ or $(a,b)=0$; observe that ψ_n preserves order and that the inductive limit is isomorphic to E as an ordered abelian group. This is how $K_0(M_\alpha)$ is obtained as an inductive limit in 2.4.

Proof. There is an increasing sequence $\{l_n\}$ and inductive systems

$$\mathbf{Z}^{l_1}\;\xrightarrow{\xi^i_1}\;\mathbf{Z}^{l_2}\;\xrightarrow{\xi^i_2}\;\mathbf{Z}^{l_3}\;\to\;\cdots$$

with limit $K_i(A)$ for $i=0,1$ (as an ordered group for $i=0$).

First, we find a homomorphism ζ_1 such that the diagram

$$\begin{array}{ccccccccc}0&\to&\mathbf{Z}^{l_1}&\xrightarrow{\iota}&\mathbf{Z}^{l_1}\oplus\mathbf{Z}^{l_1}&\xrightarrow{\rho}&\mathbf{Z}^{l_1}&\to&0\\&&\bar\xi^1_1\downarrow&&\zeta_1\downarrow&&\downarrow\bar\xi^0_1&&\\0&\to&K_1(A)&\longrightarrow&E&\longrightarrow&K_0(A)&\to&0\end{array}$$

is commutative. Set $k_1 = l_1, m(1) = 2$ and $\theta_1^i = \overline{\xi}_1^i$; suppose that we have an increasing sequence $\{k_1, \dots, k_n\}$ of positive integers and commutative diagrams:

(3)

$$
\begin{array}{ccccccccccc}
 & & \mathbf{Z}^{l_{m(1)}} & \longrightarrow & \mathbf{Z}^{l_{m(2)}} & \longrightarrow & \cdots & & \longrightarrow & \mathbf{Z}^{l_{m(n)}} \to & \cdots \\
 & \nearrow\gamma_1^i & \delta_1^i\downarrow & \nearrow\gamma_2^i & \delta_2^i\downarrow & & \cdots & & \nearrow\gamma_n^i & & \\
\mathbf{Z}^{k_1} & \xrightarrow{\chi_1^i} & \mathbf{Z}^{k_2} & \xrightarrow{\chi_2^i} & \mathbf{Z}^{k_3} & \longrightarrow & \cdots & \longrightarrow & \mathbf{Z}^{k_n} & &
\end{array}
$$

and

$$
\begin{array}{ccccccc}
\mathbf{Z}^{k_1}\oplus\mathbf{Z}^{k_1} & \xrightarrow{\psi_1} & \mathbf{Z}^{k_2}\oplus\mathbf{Z}^{k_2} & \xrightarrow{\psi_2} & \cdots & \to & \mathbf{Z}^{k_n}\oplus\mathbf{Z}^{k_n} \\
\downarrow & & \downarrow & & \cdots & & \downarrow \\
E & = & E & = & \cdots & = & E
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \to & \mathbf{Z}^{k_j} & \xrightarrow{\iota} & \mathbf{Z}^{k_j}\oplus\mathbf{Z}^{k_j} & \xrightarrow{\rho} & \mathbf{Z}^{k_j} & \to & 0 \\
 & & \theta_j^1\downarrow & & \zeta_j\downarrow & & \downarrow\theta_j^0 & & \\
0 & \to & K_0(A) & \longrightarrow & E & \longrightarrow & K_1(A) & \to & 0
\end{array}
$$

for $j = 1, 2, \dots, n$, where ψ_j is of the form $\psi_j(a,b) = (\chi_j^1(a) + \psi_j^1(b), \chi_j^0(b))$, and $\theta_j^i = \overline{\xi}_{m(j)}^i \circ \gamma_j^i$.

Choose $a_i \in E$ such that $q(a_i) = \overline{\xi}_{m(n)}^0(e_i)$. By the commutativity of the diagram

$$
\begin{array}{ccccc}
\mathbf{Z}^{k_n}\oplus\mathbf{Z}^{k_n} & \xrightarrow{\rho} & \mathbf{Z}^{k_n} & \xrightarrow{\gamma_n^0} & \mathbf{Z}^{l_{m(n)}} \\
\zeta_n\downarrow & & \theta_n^0\downarrow & & \downarrow\overline{\xi}_{m(n)}^0 \\
E & \xrightarrow{q} & K_0(A) & = & K_0(A)
\end{array}
$$

we have for $i = 1, \dots, k_n$

$$\zeta_n(0, e_i) - \sum_j \gamma_n^0(j,i) a_j \in \iota(K_1(A)).$$

Since $K_1(A) = \bigcup_n \operatorname{im} \overline{\xi}_n^1$, these elements are contained in $\iota(\operatorname{im} \overline{\xi}_{m(n+1)}^1)$ for some $m(n+1) > m(n)$. Let $k_{n+1} = l_{m(n)} l_{m(n+1)}$, and index the coordinates in $\mathbf{Z}^{k_{n+1}}$ by (s,t), $s = 1, \dots, l_{m(n+1)}$, $t = 1, \dots, l_{m(n)}$. Define $\delta_n^i : \mathbf{Z}^{l_{m(n)}} \to \mathbf{Z}^{k_{n+1}}$ by

$$\delta_n^1((s,t),t') = \begin{cases} \xi_{m(n+1)m(n)}^1(s,t) & \text{if } t = t', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_n^0((s,t),t') = \begin{cases} 1 & \text{if } t = t', \\ 0 & \text{otherwise.} \end{cases}$$

Define $\gamma_{n+1}^i : \mathbf{Z}^{k_{n+1}} \to \mathbf{Z}^{l_{m(n+1)}}$ by

$$\gamma_{n+1}^1(s',(s,t)) = \begin{cases} 1 & \text{if } s = s', \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\gamma_{n+1}^0(s',(s,t)) = \begin{cases} \xi_{m(n+1)m(n)}^0(s,t) & \text{if } s = s', \\ 0 & \text{otherwise.} \end{cases}$$

These are defined so that $\gamma_{n+1}^i \circ \delta_n^i = \xi_{m(n+1)m(n)}$, the quotient $\mathbf{Z}^{k_{n+1}} / \operatorname{im} \delta_n^0$ has no torsion, and γ_{n+1}^1 is surjective. Diagram (3) may now be extended in the following way:

$$
(4)\qquad \begin{array}{ccccccccc}
 & \cdots & & \longrightarrow & \mathbf{Z}^{l_{m(n)}} & \longrightarrow & \mathbf{Z}^{l_{m(n+1)}} & \longrightarrow & \cdots \\
 & \cdots & & \nearrow\gamma_n^i & \delta_n^i\downarrow & \nearrow\gamma_{n+1}^i & & & \\
\cdots & \longrightarrow & \mathbf{Z}^{k_n} & \xrightarrow{\chi_n^i} & \mathbf{Z}^{k_{n+1}} & & & &
\end{array}
$$

where $\chi_n^i = \delta_n^i \gamma_n^i$.

Since $\delta_n^0(e_t) = \sum_s e_{(s,t)}$ and thus $\overline{\xi}^0_{m(n)}(e_t) = \theta^0_{n+1}(\sum_s e_{(s,t)})$ (where $\theta^i_{n+1} = \overline{\xi}^i_{m(n+1)} \circ \gamma^i_{n+1}$), there exist $a_{(s,t)} \in E$ such that $a_t = \sum_s a_{(s,t)}$ and $q(a_{(s,t)}) = \theta^0_{n+1}(e_{(s,t)})$. Since

$$\overline{\xi}^1_{m(n+1)}(\mathbf{Z}^{l_{m(n+1)}}) = \theta^1_{n+1}(\mathbf{Z}^{k_{n+1}})$$

and

$$\chi_n^0((s,t),i) = \delta_n^0((s,t),t)\gamma_n^0(t,i) = \gamma_n^0(t,i),$$

we obtain

$$\zeta_n(0,e_i) - \sum_t \gamma_n^0(t,i) \sum_s a_{(s,t)} = \zeta_n(0,e_i) - \sum_{s,t} \chi_n^0((s,t),i) a_{(s,t)} \in \theta^1_{n+1}(\mathbf{Z}^{k_{n+1}}).$$

Thus there exist $(\psi_n^1((s,t),i))$ such that the left hand side equals

$$\sum_{s,t} \psi_n^1((s,t),i)\theta^1_{n+1}(e_{(s,t)}).$$

Define $\psi_n : \mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n} \to \mathbf{Z}^{k_{n+1}} \oplus \mathbf{Z}^{k_{n+1}}$ by

$$\psi_n(a,b) = (\chi_n^1(a) + \psi_n^1(b), \chi_n^0(b))$$

and define $\zeta_{n+1} : \mathbf{Z}^{k_{n+1}} \oplus \mathbf{Z}^{k_{n+1}} \to E$ by

$$\zeta_{n+1}(e_{(s,t)}, e_{(s',t')}) = \iota \circ \theta^1_{n+1}(e_{(s,t)}) + a_{(s',t')}.$$

Then one checks that

$$
\begin{array}{ccc}
\mathbf{Z}^{k_n} \oplus \mathbf{Z}^{k_n} & \xrightarrow{\psi_n} & \mathbf{Z}^{k_{n+1}} \oplus \mathbf{Z}^{k_{n+1}} \\
\zeta_n \downarrow & & \zeta_{n+1}\downarrow \\
E & = & E
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \to & \mathbf{Z}^{k_{n+1}} & \xrightarrow{\iota} & \mathbf{Z}^{k_{n+1}} \oplus \mathbf{Z}^{k_{n+1}} & \xrightarrow{\rho} & \mathbf{Z}^{k_{n+1}} & \to & 0 \\
 & & \theta^1_{n+1}\downarrow & & \zeta_{n+1}\downarrow & & \downarrow \theta^0_{n+1} & & \\
0 & \to & K_1(A) & \longrightarrow & E & \longrightarrow & K_0(A) & \to & 0
\end{array}
$$

are commutative.

For example, since

$$
\begin{aligned}
\psi_n(e_i, 0) &= \left(\sum_{s,t} \chi_n^1((s,t),i) e_{(s,t)}, 0 \right), \\
\chi_n^1((s,t),i) &= \xi^1_{m(n+1)m(n)}(s,t)\gamma_n^1(t,i), \\
\theta^1_{n+1}(e_{(s,t)}) &= \overline{\xi}^1_{m(n+1)}(e_s),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\zeta_{n+1} \circ \psi_n(e_i, 0) &= \sum_{s,t} \chi_n^1((s,t),i)\iota \circ \theta^1_{n+1}(e_{(s,t)}) = \sum_t \gamma_n^1(t,i)\iota \circ \overline{\xi}^1_{m(n)}(e_t) \\
&= \iota \circ \overline{\xi}^1_{m(n)} \circ \gamma_n^1(e_i) = \iota \circ \theta_n^1(e_i) = \zeta_n(e_i, 0). \quad \square
\end{aligned}
$$

3.11. *Proof of the theorem for the case* $i = 0$**.** By Lemma 3.10 we may assume that the extension

$$0 \;\rightarrow\; K_1(A) \;\rightarrow\; E \;\rightarrow\; K_0(A) \;\rightarrow\; 0$$

is obtained as the inductive limit of the sequence

$$\begin{array}{ccccccccccc}
0 & \rightarrow & \mathbf{Z}^{k_1} & \xrightarrow{\iota} & \mathbf{Z}^{k_1} & \oplus & \mathbf{Z}^{k_1} & \xrightarrow{\rho} & \mathbf{Z}^{k_1} & \rightarrow & 0 \\
 & & \chi_1^1\downarrow & & \chi_1^1\downarrow & \swarrow\psi_1^1 & \downarrow\chi_1^0 & & \downarrow\chi_1^0 & & \\
0 & \rightarrow & \mathbf{Z}^{k_2} & \xrightarrow{\iota} & \mathbf{Z}^{k_2} & \oplus & \mathbf{Z}^{k_2} & \xrightarrow{\rho} & \mathbf{Z}^{k_2} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & \swarrow & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots
\end{array}$$

By passing to a subsequence we may assume that $M(\chi_{n+1}^0) \geq \max(2L(\chi_n^1), 4)$; this will ensure that the inductive limit algebra to be constructed has real rank zero. As in 1.6, we construct an inductive system for A with

$$A_n = \bigoplus_{k=1}^{k_n} M_{[n,k]} \otimes C(\mathbf{T})$$

and $\varphi_n : A_n \to A_{n+1}$ so that φ_n induces $\chi_n^i : K_i(A_n) \to K_i(A_{n+1})$ and the partial embeddings are standard embeddings of type $(\chi_n^0(i,j), \chi_n^1(i,j))$.

We define α using a sequence of unitaries $u_n \in A_n$ chosen recursively such that

$$[\varphi_n(p_{n,j})u_{n+1}p_{n+1,i}\,\varphi_n(u_n^* p_{n,j})]_1 = [n,j]\psi_n^1(i,j),$$

where $[\,\cdot\,]_1$ denotes the class of a unitary in

$$K_1(\varphi_n(p_{n,j})A_{n+1,i}\varphi_n(p_{n,j})) \cong \mathbf{Z}$$

(see 2.3). Set $q_{ij}^{n+1} = \varphi_n(p_{n,j})p_{n+1,i}$ and put

$$\begin{aligned} A_{ij}^{n+1} &= q_{ij}^{n+1}A_{n+1}q_{ij}^{n+1}, \\ B_{ij}^{n+1} &= q_{ij}^{n+1}\varphi_n(A_n)' \cap A_{n+1}q_{ij}^{n+1}; \end{aligned}$$

one checks that, under the identification $K_1(A_{ij}^{n+1}) \cong \mathbf{Z}$, the range of the map $K_1(B_{ij}^{n+1}) \to K_1(A_{ij}^{n+1})$ (induced by the embedding $B_{ij}^{n+1} \subset A_{ij}^{n+1}$) is $[n,j]\mathbf{Z}$. Set $u_1 = 1$; given u_n, construct u_{n+1} as follows. Choose a unitary $w_{ij}^{n+1} \in B_{ij}^{n+1} \subset A_{ij}^{n+1}$ such that $[w_{ij}^{n+1}]_1 = [n,j]\psi_n^1(i,j)$. Set $u_{n+1} = (\bigoplus w_{ij}^{n+1})u_n$; since $\bigoplus w_{ij}^{n+1} \in A_{n+1} \cap \varphi_n(A_n)'$, one has $\alpha_{n+1} \circ \varphi_n = \varphi_n \circ \alpha_n$ (where $\alpha_n = \operatorname{Ad} u_n$). Then $\alpha = \lim \operatorname{Ad} \overline{\varphi}_n(u_n)$ has the desired properties. □

3.12. Corollary. *Let A be a simple unital $A\mathbf{T}$ algebra of real rank zero and $g \in KK(A,A)$ such that $\gamma(g) = 0$. Then there is an approximately inner automorphism α of A such that $g = 1 - [\alpha]$.*

Proof. This follows immediately from the preceding theorem and the universal coefficient theorem (see [RS, 1.17]). □

3.13. Corollary. *Let A and B be simple unital $A\mathbf{T}$ algebras of real rank zero and g be an invertible element in $KK(A,B)$ such that $\gamma(g)$ preserves positivity and the class of the unit. Then there is an isomorphism $\varphi : A \to B$ such that $g = [\varphi]$.*

Proof. Since g is invertible, $\gamma(g)_i \in \operatorname{Hom}(K_i(A), K_i(B))$ is an isomorphism for $i = 0, 1$; further, $\gamma_0([1_A]) = [1_B]$ and $\gamma_0(K_0(A)^+) \subset K_0(B)^+$. Hence, by Elliott's classification theorem (cf. [E, 7.1]) there is an isomorphism, $\psi : A \to B$, such that $\gamma(g) = \gamma([\psi])$. By the preceding corollary it follows that there is an approximately inner automorphism α such that $g = [\psi \circ \alpha]$; set $\varphi = \psi \circ \alpha$. ☐

3.14. *Remark.* Let $\operatorname{Inn}(A)$ denote the group of inner automorphisms of a C*-algebra A and let $\overline{\operatorname{Inn}}(A)$ denote its closure, the group of approximately inner automorphisms of A; further, let $\overline{\operatorname{Inn}}_0(A)$ denote the subgroup of approximately inner automorphisms which can be approximated by automorphisms of the form $\operatorname{Ad} u$ with u in the connected component of the unitary group. In [ER, 4.5] Elliott and Rørdam show that for A a simple unital $A\mathbf{T}$ algebra of real rank zero one has,

$$\overline{\operatorname{Inn}}(A)/\overline{\operatorname{Inn}}_0(A) \cong \varprojlim K_1(A)/n_jK_1(A) \tag{5}$$

where n_j ranges over the directed set of divisors of [1] in $K_0(A)$ (we assume that $n_j | n_{j+1}$) and the isomorphism commutes with the canonical homomorphism from $K_1(A)$ to both groups. They ask whether $\overline{\operatorname{Inn}}_0(A)$ must be arc-connected. By construction the α we use in the proof of Theorem 3.1 for the case $i = 1$ (see 3.8) is in $\overline{\operatorname{Inn}}_0(A)$. Thus, to answer the question in the negative it suffices to find a simple unital $A\mathbf{T}$ algebra of real rank zero for which $\operatorname{Ext}(K_1(A), K_0(A)) \neq 0$; there is then an $\alpha \in \overline{\operatorname{Inn}}_0(A)$ for which $\eta_1(\alpha) \neq 0$ (note that η_1 is a homotopy invariant). It is also possible that $\eta_0(\alpha) \neq 0$ for such α, as shown below.

An outer form of the Elliott-Rørdam invariant for $\alpha \in \overline{\operatorname{Inn}}(A)$ may be obtained from $\eta_0(\alpha)$ as follows: taking the quotient of both sides of the above isomorphism (5) by the image of $K_1(A)$, one obtains

$$\begin{aligned}(\overline{\operatorname{Inn}}(A)/\operatorname{Inn}(A))/(\overline{\operatorname{Inn}}_0(A)/\overline{\operatorname{Inn}}_0(A) \cap \operatorname{Inn}(A)) &\cong (\varprojlim K_1(A)/n_jK_1(A))/K_1(A) \\ &\cong \varprojlim{}^1 n_jK_1(A).\end{aligned}$$

With n_j as above, let D denote the inductive limit $\lim D_j$, where $D_j = \mathbf{Z}$ and the map $D_j \to D_{j+1}$ is given by multiplication by n_{j+1}/n_j ; we identify D with the divisible hull of $\mathbf{Z}[1_A]$ in $K_0(A)$ via the map $\iota : D \to K_0(A)$. It follows that

$$\varprojlim{}^1 n_jK_1(A) \cong \varprojlim{}^1 \operatorname{Hom}(D_j, K_1(A)) \cong \operatorname{Ext}(D, K_1(A)).$$

With identifications as above the Elliott-Rørdam invariant for α is $\iota^*(\eta_0(\alpha))$.

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